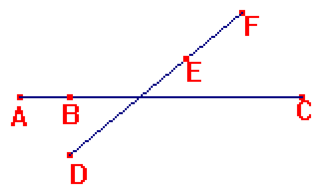
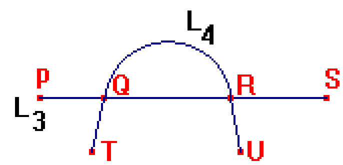
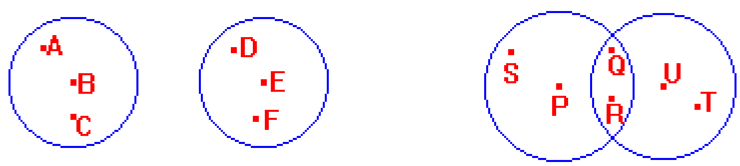
2.3 **Incidence Axioms for Geometry**

We introduce the idea of a finite geometry as a finite system satisfying the axioms of incidence. This is a way to create models for the axioms in order to assure consistency. We use diagrams that are mere representations of abstract systems, thus lines $\{A, B, C\} \cap \{D, E, F\} = \emptyset$, while $l_3 \cap l_4 = \{Q, R\}$.





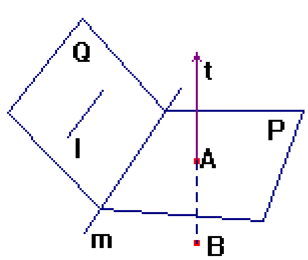
Clearly we could also have used Venn diagrams.



Remarks. 1. These axioms are an adaptation of Hilbert's axioms, to stablish how points, lines, and planes interact.

2. To facilitate the statement & use of this axioms we use basic set theory notation $x \in S$, $A \subseteq T$, $A \cup B$, $A \cap B$, $S =$ Universal set (all points in space)

3. Undifined terms: point, line, plane, and space.



Notation:

$l \subset$ Q — "l lies on Q"

$t \cap P = \{A\}$ — "t meets P"

$l \parallel P$ — "l is parallel to P"

$P \cap Q = m$ — "P meets Q in m"

$l \cap m = \phi$

$\{A\}\&\{B\}$ are collinear

Axiom I-1: Each two distinct points P & Q determine a line $\overleftrightarrow{PQ}$.

Hence P, Q $\in \overleftrightarrow{PQ}$, $\{P,Q\} \subset \overleftrightarrow{PQ}$

This axiom can be restated as follows:

Theorem 1. If $C \in \overleftrightarrow{AB}$, $D \in \overleftrightarrow{AB}$ and $C \neq D$, then $\overleftrightarrow{CD} = \overleftrightarrow{AB}$

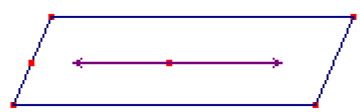
Axiom I-2: Three noncollinear points determine a plane

Ex. a) Tripod. b) A 3-legged table
(always stable over a plane).

Axiom I-3:

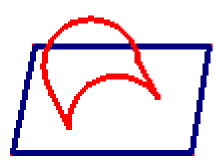
If two points lie in a plane, then any line determined by those two points lies in that plane.

That is, lines are straight.



Axiom I-4: If two planes meet, their intersection is a line

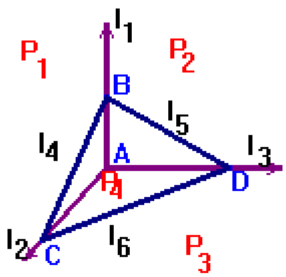
Planes are flat



Axiom I-5:

Space consists of at least 4 non coplanar points, and contains 3 noncollinear points. Each plane contains at least 3 noncollinear points, and each line contains at least 2 distinct points

Model for Axioms I.1-I.5:



(With only the 4 points guaranteed by Ax. I.5) Explore whether the axioms hold in the given figure. List all possible lines and planes in this geometry, and test if each axiom is satified. Can you generalize?

Solution. 1. Lines: $l_1 = \{A, B\}, l_2 = \{A, C\}, l_3 = \{A, D\}$, $l_4 = \{B, C\}, l_5 = \{B, D\}, l_6 = \{C, D\}$. $\binom{4}{2} = 6$ possible lines $\Rightarrow$ Axiom I-1 holds.

2. Planes: $P_1 = \{A, B, C\}$, $P_2 = \{A, B, D\}, P_3 = \{A, C, D\}$, $P_4 = \{B, C, D\}$. $\binom{4}{3} = 4$ possible planes $\Rightarrow$ Axiom I-2 holds.

3. Pairs of points:

$l_1, l_2, l_4 \in P_1; l_1, l_3, l_5 \in P_2$ hence Aiom I-3
$l_2, l_3, l_6 \in P_3; l_4, l_5, l_6 \in P_4$ holds

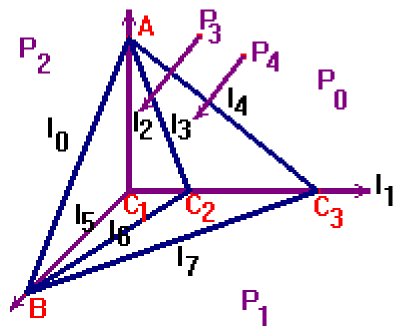
4. $P_1 \cap P_2 = l_1;\ P_1 \cap P_3 = l_2;\ P_1 \cap P_4 = l_4;\ P_2 \cap P_3 = l_3;$ $P_2 \cap P_4 = l_5;\ P_3 \cap P_4 = l_6,\ \Rightarrow$ Axiom I-4 holds.

5. Axiom I-5 holds.

6. Let the *n* points be $A, B, C_1, \ldots, C_{n-2}$. For $k = 1, 2, \ldots, n$ take $l_0 = \{A, B\}, l_1 = \{C_1, C_2, \ldots, C_n\}, l_{k+1} = \{A, C_k\}$, $l_{n+k+1} = \{B, C_k\}$, $P_0 = l_1 \cup \{A\}, P_1 = l_1 \cup \{B\}$, and $P_{k+1} = \{A, B, C_k\}$.

Example for A, B, C_1, C_2, C_3 :

$l_0 = \{A, B\}, l_1 = \{C_1, C_2, C_3\},$
$l_2 = \{A, C_1\}, l_3 = \{A, C_2\},$
$l_4 = \{A, C_3\}, l_5 = \{B, C_1\},$
$l_6 = \{B, C_2\}, l_7 = \{B, C_3\}$
$P_0 = l_1 \cup \{A\}, P_1 = l_1 \cup \{B\},$
$P_2 = \{A, B, C_1\}, P_3 = \{A, B, C_2\},$
$P_4 = \{A, B, C_3\}.$

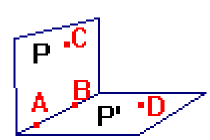


Ex. Prove explicitly from the incidence axioms that i) a plane cannot be a line, & ii) each line is contained by at least two planes whose intersection is that line.

Sol. i) By axiom I-5, each plane contains at least 3 non collinear points, this would contradict that a plane is a line.

ii) CONCLUSIONS	JUSTIFICATIONS
(1) Let $\overleftrightarrow{AB}$ be a line	Given
(2) Let C be a point not on $\overleftrightarrow{AB}$	Axiom I-5
(3) Let P be the plane determined by A,B,&C	Axiom I-2
(4) Let D $\notin P$	Axiom I-5
(5) Let P' be the plane determined by A,B,&D	Axiom I-2
(6) A, B $\in P \cap P'$	(3) & (5)
(7) $\overleftrightarrow{AB} = P \cap P'$	Axioms I-4 & I-1

$\therefore$ $\overleftrightarrow{AB}$ is contained in at least two planes



Theorem 2.

i) If two distinct lines l and m meet, their intersection is a single point.

ii) If a line meets a plane and is not contained by that plane, their intersection is a point.

Proof

I. Given: Distinct lines l and m, point A on $l \cap m$.
Prove: No other point, besides A, lies in $l \cap m$.

CONCLUSIONS	JUSTIFICATIONS
(1) Assume false. Assume pt. B $\neq$ A lies on both l and m.	assumption for indirect proof
(2) A and B belong to both l and m	given and Step (1)
(3) $l = m$ $\rightarrow\leftarrow$	Axiom I-1
(4) $\therefore$ A is the only point on both l and m	Rule of Elimination

II. Given: Line l not lying in plane P, A $\in l \cap P$.
Prove: No other point, besides A, lies in $l \cap P$.

CONCLUSIONS	JUSTIFICATIONS
(1) Assume false, i.e., B $\neq$ A $\wedge$ B $\in l \cap P$	Assump. for Ind. proof
(2) $l = \overleftrightarrow{AB}$	Hyp. & Thm. 1
(3) $l \subseteq P$ $\rightarrow\leftarrow$	Ax I-3
(4) $\therefore$ A is the only point on both l & P	