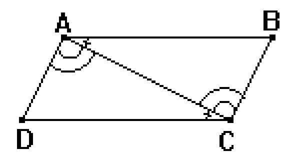
4.2 Parallelograms and Trapezoids: Parallel Projection

<u>*Definition*</u> A convex quadrilateral $\diamond$ABCD is called a **parallelogram** if the opposite sides $\overline{AB}$, $\overline{CD}$ and $\overline{BC}$, $\overline{AD}$ are parallel. A **rhombus** is a parallelogram having two adjacent sides congruent. A **square** is a rhombus having two adjacent sides perpendicular.

<u>*Theorem 1*</u> A diagonal of a parallelogram divides it into two congruent triangles.

Proof: We want to show that $\triangle ABC \simeq \triangle CDA$.

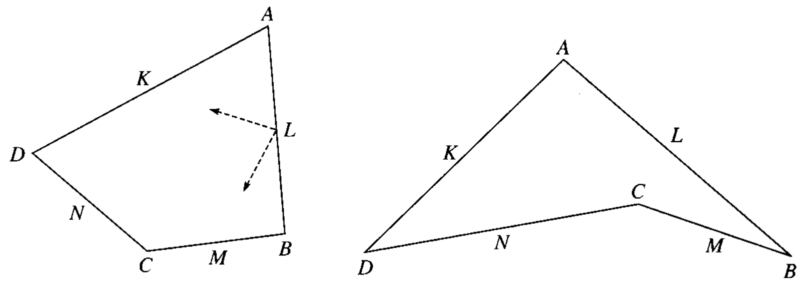
1) $\angle BAC \simeq \angle DCA$ (Z-Property of Parallelism)
2) $\angle ACB \simeq \angle CAD$ (Z-Property of Parallelism)
3) AC = AC (Reflexive Property of =)
4) $\therefore \triangle ABC \simeq \triangle CDA$ (ASA)



<u>*Corollaries*</u>:

- The opposite sides of a parallelogram are congruent.
- The diagonals of a parallelogram bisect each other.
- If a convex quadrilateral has opposite sides congruent, then it is a parallelogram.
- If a convex quadrilateral has a single pair of opposite sides which are both parallel and congruent, it is a parallelogram.
- A parallelogram is a rhombus iff its diagonals are perpendicular
- A parallelogram is a rectangle iff its diagonals are congruent.
- A parallelogram is a square iff its diagonals are both congruent and perpendicular.

Moment for Discovery: A Quadrilateral Within a Quadrilateral



1. Draw an arbitrary quadrilateral ◇ABCD, one convex, and one nonconvex.

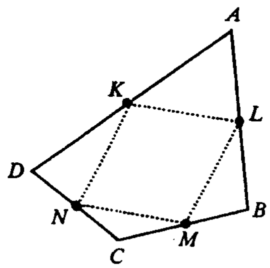
2. Locate, as carefully as you can, or by actual construction, the midpoints K, L, M, and N of the respective sides.

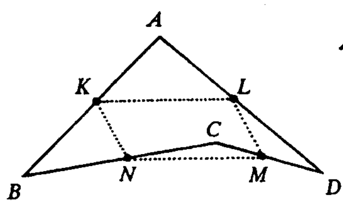
3. Join the points K, L, M, and N in order, forming a quadrilateral. Do you observe anything in particular?

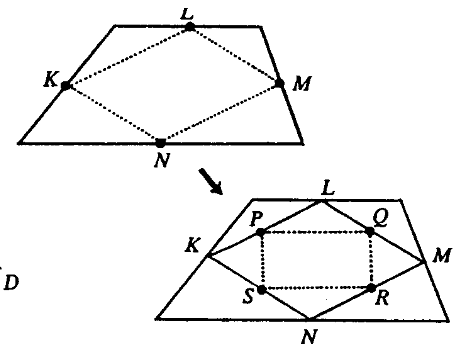
4. Draw a kite ◇ABCD (with AB = AD and BC = CD), and locate the midpoints K, L, M, and N as before. Join these points, as before, forming ◇KLMN. Do you observe anything different? Can you prove what you observed?

5. Draw isosceles trapezoid ◇ABCD (see definition which follows) and draw ◇KLMN determined as before. Go one step further, and draw ◇PQRS, where P, Q, R, and S are the midpoints of the sides of ◇LKMN. Do you observe anything? Can you prove it?

6. What theorems have you discovered? Write a concise statement of each.



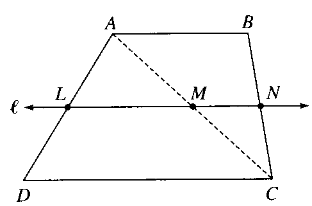




In 1-4, observe that ◇ KLMN is a parallelogram in both cases, making use of the Midpoint Connector Theorem. (In 4, ◇ KLMN is a rectangle). In 5 ◇ KLMN is a rhombus and ◇ PQRS is a rectangle.

<u>Definition</u>: A **trapezoid** is a (convex) quadrilateral with at least two opposite sides parallel, called the **bases**, with the other two sides called the **legs**. The segment joining the midpoints of the legs of a trapezoid is called the **median** (also the term used for the line passing through those midpoints). A trapezoid is said to be **isosceles** iff its legs are congruent and it is not a parallelogram.

<u>Theorem 2</u>: Midpoint Connector Theorem for Trapezoids
If a line segment bisects one leg of a trapezoid and is parallel to the base, then it is the median and its length is one-half the sum of the lengths of the two bases. Conversely, the median of a trapezoid is parallel to the bases.



Proof: Let line l bisect leg $\overline{AD}$ at L and be parallel to base $\overline{CD}$; hence $l \parallel \overleftrightarrow{AB}$ by definition of trapezoid and Transitivity of Parallelism. Construct diagonal $\overline{AC}$.

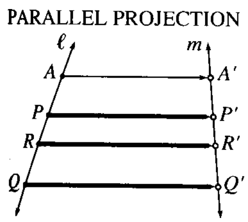
1) l bisects $\overline{AC}$ at some point M. (cor. to Midpoint Connector thm.)

2) $\therefore l$ bisects $\overline{\text{BC}}$ at some point N (same reason)

3) LM = $\frac{1}{2}$DC and MN = $\frac{1}{2}$AB (Midpoint Connector Theorem)

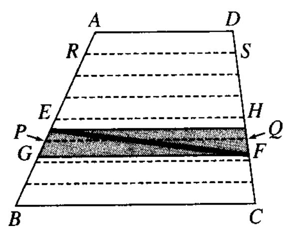
4) L-M-N (ray $\overrightarrow{\text{AC}} \subseteq$ interior $\angle$BAD)

5) $\therefore$ LN = LM + MN = $\frac{1}{2}$(AB + DC) (Steps (3) and (4))



Given the lines l and m and the points A $\in l$ and A' $\in m$, we define the 1-1 correspondence: P$\longleftrightarrow$P' if $\overleftrightarrow{\text{PP'}} \parallel$ AA'. **Betweenness** and **ratios of segments** are **invariant properties** of this mapping.

<u>*Lemma:*</u> If $\overleftrightarrow{\text{EF}}$ is not parallel to $\overleftrightarrow{\text{BC}}$, then AE/AB $\neq$ DF/DC.



Proof: Construct the parallels $\overleftrightarrow{\text{EH}}$ and $\overleftrightarrow{\text{GF}}$ to $\overleftrightarrow{\text{BC}}$ through E and F, respectively. This forms either a Z or backward Z depending on whether A-G-E or A-E-G. Let's assume for sake of argument that A-E-G holds. Hence, since betweeness is preserved under parallel projection, D-H-F.

1) Bisect segments $\overline{\text{AB}}$ and $\overline{\text{DC}}$, then bisect the segments which these midpoints determine, and continue the bisection process indefinitely. It is clear that at some stage we will have one of these midpoints, say P, falling on segment $\overline{\text{EG}}$, and, correspondingly, midpoint Q falls on $\overline{\text{HF}}$.

2) The lines joining corresponding midpoints are parallel to $\overleftrightarrow{\text{BC}}$ and $\overleftrightarrow{\text{AD}}$. (Midpoint Connector Theorem for Trapezoids)

3) For some integer k, AP = k · AR = k · $\frac{AB}{2^n}$ (n = number of bisections required). Since betweeness is preserved, DQ = k · DS = k · $\frac{DC}{2^n}$.

4) $\therefore \frac{AP}{DQ} = \frac{AB}{DC}$, or $\frac{AP}{AB} = \frac{DQ}{DC}$ (algebra)

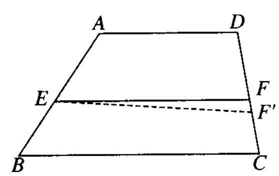
5) Also, by betweenness, AE $<$ AP and DQ $<$ DF, so
$\frac{AE}{AB} < \frac{AP}{AB} = \frac{DQ}{DC} < \frac{DF}{DC}$

6) The case A-G-E leads to the result $\frac{AE}{AB} > \frac{DF}{DC}$.

7) $\therefore \frac{AE}{AB} \neq \frac{DF}{DC}$

<u>*Corollary*</u> If AE/AB = DF/DC, then $\overleftrightarrow{EF} \parallel \overleftrightarrow{BC}$.

<u>*Theorem 3*</u>: Parallel Projection Theorem
The mapping from l to m described previously preserves ratios of line segments.



Proof: We need to prove that if $\overleftrightarrow{EF} \parallel \overleftrightarrow{BC}$, then AE/AB = DF/DC. Locate F' on $\overline{DC}$ such that DF' = DC · (AE/AB), or AE/AB = DF'/DC, and construct line $\overleftrightarrow{EF'}$. By the preceding corollary, $\overleftrightarrow{EF'} \parallel \overleftrightarrow{BC}$. But $\overleftrightarrow{EF} \parallel \overleftrightarrow{AD}$ by hypothesis, hence $\overleftrightarrow{EF} = \overleftrightarrow{EF'}$. (Why?) Therefore, F' = F and AE/AB = DF/DC, as desired.

Corollary: The Side-Splitting Theorem

If a line parallel to the base BC of $\triangle$ABC cuts the other two sides AB and AC at E and F, respectively, then $\frac{AE}{AB} = \frac{AF}{AC}$, and by algebra $\frac{AE}{EB} = \frac{AF}{FC}$.

4.3 Similar Triangles, Pythagorean Theorem, Trigonometry

Definition: Two polygons P_1 and P_2 are said to be similar, denoted $P_1 \sim P_2$, iff under some correspondence of their vertices, corresponding angles are congruent, and the ratio of the lengths of corresponding sides is constant (= k). The number k is called the constant of proportionality, or scale factor, for the similarity.

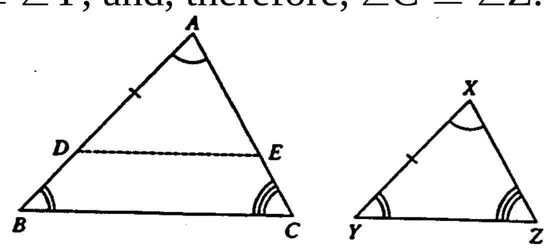
Similar Polygons:

$\triangle ABC \sim \triangle XYZ$ iff $\angle A \simeq \angle X$, $AB = k \cdot XY$
$\angle B \simeq \angle Y$, $BC = k \cdot YZ$
$\angle C \simeq \angle Z$, $AC = k \cdot XZ$

Remarks: 1) If k = 1 the triangles are congruent.
2) Similarity $\sim$ "same shape,"
congruent $\sim$ "same shape and size".

Moment for Discovery: Exploring a Criterion for Similar Triangles

Suppose that under the correspondence ABC⟷XYZ we have $\angle A \simeq \angle X$, $\angle B \simeq \angle Y$, and, therefore, $\angle C \simeq \angle Z$.



1. If AB = XY, what must be true regarding the lengths of the remaining two sides of the triangles?

2. Assume AB > XY. Locate D on $\overline{AB}$ such that AD = XY. What will make AD/AB = AE/AC for E on $\overline{AC}$?

3. Is $\triangle XYZ \simeq \triangle ADE$? Can you see how to prove XY/AB = XZ/AC for this case?

4. What about the case $AB < XY$? What is your conclusion in general? (State this as a general lemma.)

5. Does your lemma imply that XZ/AC = YZ/BC?

6. What have you proven now, in general?

Solution:
1. BC = YZ and AC = XZ (the triangles would be congruent by ASA).

2. If line $\overleftrightarrow{DE}$ were parallel to line $\overleftrightarrow{BC}$.

3. Yes, by ASA. By the Parallel Projection Theorem.
$AD/AB = AE/AC \Rightarrow XY/AB = XZ/AC$ or $AB/XY = AC/XZ$

4. LEMMA. *If two triangles have the three angles of one congruent, respectively, to the three angles of the other, then pairs of corresponding sides are the same ratio.*

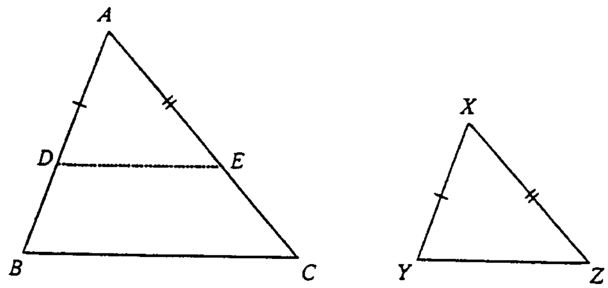
5. Yes, which implies, by algebra, AB = kXY, BC = kYZ, and AC = kXZ, where k = AB/XY. Hence, $\triangle ABC \sim \triangle XYZ$.

It follows that:

Theorem 1: AA SIMILARITY CRITERION
If, under some correspondence, two triangles have two pairs of corresponding angles congruent, the triangles are similar under that correspondence.

Theorem 2: SAS SIMILARITY CRITERION
If in $\triangle ABC$ and $\triangle XYZ$ we have AB/XY = AC/XZ and $\angle A \simeq \angle X$, then $\triangle ABC \sim \triangle XYZ$.

Proof:



Given: AB = kXY, AC = kXZ, and $\angle A \simeq \angle X$.
Prove: $\triangle ABC \sim \triangle XYZ$.

The result follows by the AA Similarity Criterion if we show that $\angle B \simeq \angle Y$. We can assume that $k < 1$ (if k = 1 the triangles are congruent; the case $k > 1$ is logically equivalent to $k < 1$).

1) On $\overline{AB}$ and $\overline{AC}$ construct $\overline{AD} \simeq \overline{XY}$ and $\overline{AE} \simeq \overline{XZ}$; draw $\overline{DE}$.

2) $\triangle ADE \simeq \triangle XYZ$ (SAS Congruence Criterion).

3) AD/AB = XY/AB = 1/k = XZ/AC = AE/AC.

4) $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$ (Corollary to Lemma of §4.2).

5) $\angle B \simeq \angle ADE \simeq \angle Y$ (F-property and CPCF).

6) $\therefore \triangle ABC \sim \triangle XYZ$ (AA Similarity)

<u>*Theorem 3*</u>: SSS SIMILARITY CRITERION
If in $\triangle ABC$ and $\triangle XYZ$ we have AB/XY = BC/YZ = AC/XZ, then $\angle A \simeq \angle X$, $\angle B \simeq \angle Y$, $\angle C \simeq \angle Z$, and $\triangle ABC \sim \triangle XYZ$.

Proof: Suppose AB = kXY, BC = kYZ, and AC = kXZ. If k > 1 locate D on $\overline{AB}$ and E on $\overline{AC}$ such that AD = XY and AE = XZ. Draw $\overline{DE}$. (Here it is not immediately clear that $\triangle$ADE $\simeq$ $\triangle$XYZ.)

1) $\triangle$ADE $\sim$ $\triangle$ABC (SAS Similarity Criterion).

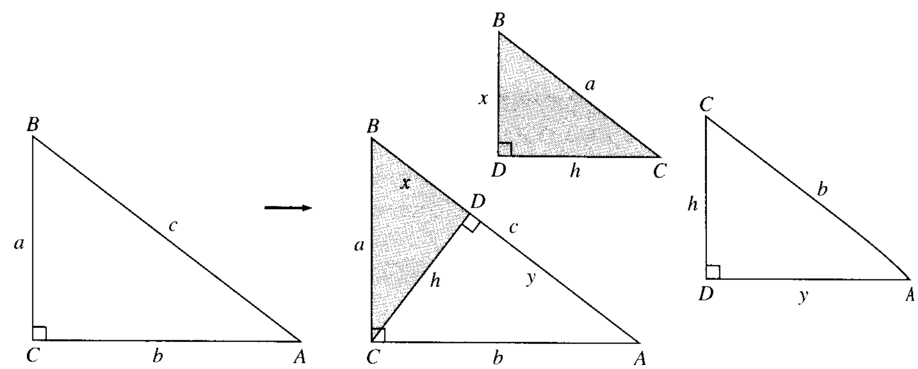
2) DE/BC = AD/AB = XY/AB = 1/k = YZ/BC $\Rightarrow$ DE = YZ.

3) $\triangle$ADE $\simeq$ $\triangle$XYZ (SSS Congruence Criterion).

4) $\angle$A $\simeq$ $\angle$X.

5) $\triangle$ABC $\sim$ $\triangle$XYZ (SAS Similarity Criterion.)

Moment for Discovery : Applying the AA Similarity Criterion



Let $\triangle$ABC be any right triangle with right angle at C, and with side lengths a = BC, b = AC, and c = AB. Drop perpendicular CD to AB. Since the angles at A and B are acute, we must have A-D-B.

1. Show that $\triangle$BDC $\sim$ $\triangle$BCA.

2. Show that $\triangle$ADC $\sim$ $\triangle$ACB.

3. Observe the corresponding sides of the three similar triangles mentioned in Steps(1) and (2): $\frac{x}{a} = \frac{a}{c}$ and $\frac{y}{b} = ?$

4. Solve for x and y in these two equations, then use the relation x + y $=$ c, and simplify. What did you prove?

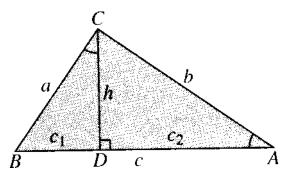
Solution: 1. m∠B = m∠B and m∠A = 90 - m∠B = m∠BCD, so by the AA Similarity Criterion, △BDC $\sim$ △BCA.

2. m∠A = m∠A and m∠B = 90 $-$ m∠A = m∠ACD, so △ADC $\sim$ △ACB.

3. $\frac{c}{b} = \frac{b}{y}$

4. x = $\frac{a^2}{c}$, y = $\frac{b^2}{c}$ $\Rightarrow$ c = x + y = $\frac{a^2}{c} + \frac{b^2}{c} = \frac{a^2+b^2}{c}$,
$\therefore c^2 = a^2 + b^2$

Example
Using similar triangles, prove the classical relationship between the altitude to the hypotenuse of a right triangle and the segments formed on the hypotenuse: $h = c_1c_2$



EUCLIDEAN TRIGONOMETRY OF THE RIGHT TRIANGLE
sin A = $\frac{a}{c}$; sin B = $\frac{b}{c}$; tan A = $\frac{a}{b}$ $a^2+b^2 = c^2$ (Pythagorean Thm)
cos A = $\frac{b}{c}$; cos B = $\frac{a}{c}$; tan B = $\frac{b}{a}$

LAW OF SINES FOR ANY TRIANGLE

$\frac{a}{sin\,A} = \frac{b}{sin\,B} = \frac{c}{sin\,C}$

LAW OF COSINES FOR ANY TRIANGLE

$a^2 = b^2 + c^2 - 2bc\,cos\,A; b^2 = a^2 + c^2 - 2ac\,cos\,B$

$c^2 = a^2 + b^2 - 2ab\,cos\,C$